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INVERSE PROBLEM OF DETERMINING THE KERNEL IN THE INTEGRO-DIFFERENTIAL BEAM VIBRATION EQUATION

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Abstract This paper addresses the problem of identifying the kernel that represents the memory of the medium in the integro-differential equation governing the forced vibrations of a beam. The problem is reduced to integral equations of the second kind of Volterra type with respect to the solution of the direct problem and the unknown kernel of the inverse problem. To solve these equations, we apply the method of compressive mappings in the space of continuous functions with exponential weight norm. The global solvability of the inverse problem and the conditional stability of the solution are established. In addition, the paper introduces an efficient numerical approach for solving the inverse problem associated with the beam equation. The method leverages the finite difference method to obtain solutions for both the displacement field $u(x,t)$ and the media viscosity coefficient $k(t)$, with the latter expressed in terms of an integral. Rigorous algorithmic development ensures accurate numerical solutions, which are validated through comparison with analytical solutions, demonstrating a high level of agreement. Furthermore, the proposed scheme is evaluated under noisy conditions, where it exhibits robustness in reducing numerical fluctuations over time. This comprehensive study highlights the reliability and effectiveness of the developed numerical approach, positioning it as a valuable tool for tackling inverse problems in structural mechanics and related fields.

Key words: inverse problem, transverse bending vibrations, kernel, Fourier method, finite difference scheme, numerical solution, noise.

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1 Introduction

Many problems involving the vibrations of rods, beams, and plates have significant applications in structural design, stability theory of rotating shafts, vibration theory of ships, and pipelines. These problems are typically described by differential equations of orders higher than the second [1], [2].

There has been a recent surge in interest regarding the study of direct and inverse problems for the vibration equation of a beam [3]–[16]. Paper [3] analyses publications and results on the dynamic behaviour of in-homogeneous beams and rods, based on foreign press materials. Direct problems with different initial and boundary conditions are investigated for the beam vibration equation in papers [4]–[7] and inverse problems for finding the coefficient are studied for them. In Paper [8], authors study the solvability of the initial boundary value problem for the general Euler-Bernoulli beam equation including also moving point loads. The numerical solutions of the beam vibration equation are given in [9]–[16]. In [15], an analytical solution of the differential equation of transverse vibrations of a piece-wise homogeneous beam in the frequency domain was obtained for various boundary conditions.

The study of inverse problems is a phenomenon that has both a long and recent history. Inverse problems in mathematical physics have been studied for many classes of differential equations. The monograph [17] focuses on inverse problems associated with the simplest equations of hyperbolic type. Methods for solving inverse dynamical problems, including proving local existence and singularity theorems, as well as numerical approaches to finding solutions, have been explored in various sources, such as [18]–[31].

2 Mathematical formulation of the inverse problem

We consider an inverse problem of determining the unknown kernel $k(t)$ and the transverse bending vibrations $u(x,t)$ of a homogeneous beam of length l that satisfy the fourth-order partial differential equation

$$u_{tt} + a^2 u_{xxxx} = \int\limits_0^t k(\tau) u(x, t-\tau) d\tau + f(x,t), \qquad (x,t) \in (0,l) \times (0,T] =: D \tag{1}$$

with the initial contions

$$u\big|_{t=0} = \varphi(x), \quad u_t\big|_{t=0} = \psi(x), \quad x \in [0,l] \tag{2}$$

and boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \quad t \in [0,T]. \tag{3}$$

In the direct problem, it is required to determine the function

$$u(x,t) \in C^{4,2}_{x,t}(D) \cap C^{2,1}_{x,t}(\overline{D}), \tag{4}$$

satisfying relations (1)–(3) with the known positive numbers a, l, T, and sufficiently smooth functions $\varphi(x)$, $\psi(x)$, $k(t)$, $f(x,t)$.

Inverse problem. Find the unknown kernel $k(t)$, if the following additional information about the solution of the direct problem (1)–(3) is available:

$$u(x_0, t) = g(t), \quad x_0 \in (0,l), \quad t \in [0,T], \tag{5}$$

where the function $g(t)$ is known.

We assume that the data of the problem (1)–(5) satisfy the following conditions:

(*A*1) $\varphi(x) \in C^7[0,l]$, $\varphi^{(j)}(0) = \varphi^{(j)}(l) = 0$, $j = 0,2,4,6$,

(*A*2) $\psi(x) \in C^5[0,l]$, $\psi^{(j)}(0) = \psi^{(j)}(l) = 0$, $j = 0,2,4$,

(*A*3) $f(x,t) \in C^{3,1}_{x,t}(\overline{D})$, $f(0,t) = f(l,t) = f''_{xx}(0,t) = f''_{xx}(l,t) = 0$, $0 \leqslant t \leqslant T$,

(*A*4) $g(t) \in C^3[0,T]$, $g(0) = \varphi(x_0) \neq 0$, $g'(0) = \psi(x_0)$, $g''(0) = f(x_0,0) - a^2\varphi^4(x_0)$.

Let $u(x,t)$ be the solution of the direct problem (1)–(4). Introducing the notation $v(x,t) = u_t(x,t)$ and differentiating (1), (3) by t, we have

$$v_{tt} + a^2 v_{xxxx} = k(t)\varphi(x) + \int\limits_0^t k(\tau) v(x, t-\tau) d\tau + F(x,t), \tag{6}$$

where $F(x,t) = f_t(x,t)$, boundary conditions

$$v(0,t) = v_{xx}(0,t) = v(l,t) = v_{xx}(l,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{7}$$

The initial conditions are easily obtained from (2) and equation (1) at $t = 0$:

$$v\big|_{t=0} = \psi(x), \quad v_t\big|_{t=0} = f(x,0) - a^2\varphi^{(4)}(x) := \phi(x), \quad x \in [0,l]. \tag{8}$$

The additional condition for the function v, as follows from (5), takes the form

$$v(x_0,t) = g'(t), \quad x_0 \in (0,l), \quad t \in [0,T]. \tag{9}$$

Thus, the inverse problem (1)–(4) is reduced to the problem of determining the same function $k(t)$ from equation (6), when the solution of this equation satisfies the equations (7)–(9). It is easy to see that if the matching conditions $\varphi(0) = \varphi''(0) = \varphi(l) = \varphi''(l) = 0$, $\psi(0) = \psi''(0) = \psi(l) = \psi''(l) = 0$, $\varphi(x_0) = g(0)$, $\psi(x_0) = g'(0)$, the inverse transformations are also true.

2.1 Study of the direct problem (6)–(8)

The solution of the direct problem (6)–(8) will be found in the following form

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) X_n(x), \tag{10}$$

where $X(x)$ is the solution to the following problem:

$$X^{(4)}(x) + \lambda X(x) = 0, \ 0 < x < l, \qquad X(0) = X''(0) = X(l) = X''(l) = 0.$$

Solving this problem we get

$$X_n(x) = \sqrt{\frac{2}{l}} \sin \mu_n x, \quad \lambda_n = -\mu_n^4 = -\left(\frac{\pi n}{l}\right)^4,$$

$$v_n(t) = \sqrt{\frac{2}{l}} \int_0^l v(x,t) \sin \mu_n x dx, \quad n = 1, 2, \ldots.$$

Formally, from (10) by term-by-term differentiation we compose the series

$$v_{tt} = \sum_{n=1}^{\infty} v_n''(t) X_n(x), \tag{11}$$

$$v_{xxxx} = \sum_{n=1}^{\infty} v_n(t) X_n^{(4)}(x) = \sum_{n=1}^{\infty} \mu_n^4 v_n(t) X_n(x). \tag{12}$$

Applying the formal scheme of the Fourier method and using (6), (7), we obtain

$$v_n''(t) + a^2\mu_n^4 v_n(t) = k(t)\varphi_n + \int_0^t k(\tau) v_n(t-\tau) d\tau + F_n(t), \quad 0 < t \leqslant T, \tag{13}$$

$$v_n(0) = \psi_n, \quad v_n'(0) = \phi_n, \quad n = 1, 2, \ldots, \tag{14}$$

where

$$\varphi_n = \int_0^l \varphi(x) X_n(x) dx, \quad \psi_n = \int_0^l \psi(x) X_n(x) dx, \quad \phi_n = \int_0^l \phi(x) X_n(x) dx,$$

$$F_n(t) = \int_0^l F(x,t)X_n(x)dx. \tag{15}$$

Using the method of [4], we present the solution of the problem (13), (14) as an integral equation

$$v_n(t) = v_{0n}(t) + \int_0^t K_n(t,\tau)v_n(\tau)d\tau, \tag{16}$$

where

$$v_{0n}(t) = \left(\psi_n \cos a\mu_n^2 t - \frac{\phi_n}{a\mu_n^2}\sin a\mu_n^2 t\right) + \frac{1}{a\mu_n^2}\int_0^t \big(\varphi_n k(s) + F_n(s)\big)\sin a\mu_n^2(t-s)ds, \tag{17}$$

$$K_n(t,\tau) = \frac{1}{a\mu_n^2}\int_\tau^t k(s-\tau)\sin a\mu_n^2(t-s)ds. \tag{18}$$

Theorem 2.1. *If there exists a function $v(x,t)$ satisfying relations (6)–(8), then it is unique.*

Based on the completeness of the system $X_n(x)$ in the space $L_2(0,l)$ we can prove the uniqueness of the solution of the problem (6)–(8). Indeed, let there exist distinct functions $v_1(x,t)$ and $v_2(x,t)$ – solutions to this problem. Then their difference $v(x,t) = v_1(x,t) - v_2(x,t)$ is the solution of the homogeneous problem (6)–(8), where $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$, $F(x,t) \equiv 0$, $\phi(x) = 0$, then $\varphi_n \equiv 0$, $\psi_n \equiv 0$, $F_n(t) \equiv 0$, $\phi_n \equiv 0$ and from (16) we obtain $v_n \equiv 0$, since v_n is a solution to the homogeneous equation:

$$v_n(t) = \int_0^t K_n(t,\tau)v_n(\tau)d\tau$$

which is equivalent to the equality

$$\int_0^l v(x,t)X_n(x)dx = 0.$$

By virtue of the completeness of the system $X_n(x)$ in the space $L_2(0,l)$, the function $v(x,t) = 0$ is almost everywhere in $[0,l]$ and at any $t \in [0,T]$. Since the function v is from the class of functions given in condition (4), v is continuous on $\overline{D}$, then $v(x,t) \equiv 0$ on $\overline{D}$. Thus, the uniqueness of the solution of the problem (6)–(8) is proved.

For each fixed n, the equation (16) is a Volterra integral equation of the second kind with respect to v_n. According to the general theory of integral equations, under proper conditions on the functions $\varphi(x)$, $\psi(x)$, $k(t)$, $f(x,t)$ it has a unique solution. The solution of this integral equation can be found by the method of successive approximations.

Further, from (17) and (18) the estimates

$$|v_{0n}(t)| \leqslant |\psi_n| + \frac{1}{a\mu_n^2}|\phi_n| + \frac{\|k\|T}{a\mu_n^2}|\varphi_n| + \frac{T}{a\mu_n^2}\|F_n\| =: C_n$$

and

$$|K_n(t,\tau)| \leqslant \frac{\|k\|(t-\tau)}{a\mu_n^2}, \quad (t,\tau) \in \triangleleft(T) = \{0 \leqslant \tau \leqslant t \leqslant T\}$$

follow, where $\|k\| = \max\limits_{t\in[0,T]} |k(t)|$, $\|F_n\| = \max\limits_{t\in[0,T]} |F_n(t)|$. Taking them into account, from (16) we get

$$|v_n(t)| \leqslant C_n + \frac{\|k\|}{a\mu_n^2} \int\limits_0^t |v_n(\tau)|(t-\tau)d\tau.$$

Hence, by virtue of Gronwall's inequality, we can conclude that the solution $v(t)$ satisfies:

$$|v_n(t)| \leqslant C_n \exp\left\{\frac{\|k\|T^2}{2a\mu_n^2}\right\}, \quad t \in [0,T], \quad n = 1,2,\dots. \tag{19}$$

Using (13) and (19), we obtain an estimate for $v_n''(t)$:

$$|v_n''(t)| \leqslant |\varphi_n|\|k\| + \|F_n\| + C_n\left(\|k\|T + a^2\mu_n^4\right)\exp\left\{\frac{\|k\|T^2}{2a\mu_n^2}\right\}.$$

Thus, the following statement is true:

Lemma 2.1. *For any $t \in [0,T]$ the following estimates*

$$|v_n(t)| \leqslant C_1\left(|\psi_n| + \frac{1}{n^2}|\phi_n| + \frac{1}{n^2}|\varphi_n| + \frac{1}{n^2}\|F_n\|\right), \tag{20}$$

$$|v_n''(t)| \leqslant C_2\left(n^4|\psi_n| + n^2|\phi_n| + n^2|\varphi_n| + n^2\|F_n\|\right), \tag{21}$$

are valid, where C_i, $i = 1,2$ are positive constants depending on T, l and $\|k\|$.

The series (10), (11) and (12) for any $(x,t) \in \overline{D}$ based on Lemma 2.1 are majorized by the series

$$C_3 \sum_{n=1}^{\infty}\left(n^4|\psi_n| + n^2|\phi_n| + n^2|\varphi_n| + n^2\|F_n\|\right), \tag{22}$$

where $C_3 > 0$ depends on a, T, l and $\|k\|$.

Since $\phi_n = f_n - a^2\varphi_n^{(4)}$, where $f_n = \int\limits_0^l f(x,0)X_n(x)dx$, $\varphi_n^{(4)} = \int\limits_0^l \varphi^{(4)}(x)X_n(x)dx$, then $|\phi_n| \leqslant |f_n| + a^2|\varphi_n^{(4)}|$. Hence, the numerical series (22) can be estimated as follows

$$C_3' \sum_{n=1}^{\infty}\left(n^4|\psi_n| + n^2\left(|\varphi_n| + |\varphi_n^{(4)}|\right) + n^2|f_n| + n^2\|F_n\|\right). \tag{23}$$

Lemma 2.2. *Under the conditions (A1)-(A3), one has the relations*

$$\psi_n = \frac{1}{\mu_n^5}\psi_n^{(5)}, \quad \varphi_n = -\frac{1}{\mu_n^3}\varphi_n^{(3)}, \quad \varphi_n^{(4)} = -\frac{1}{\mu_n^3}\varphi_n^{(7)}, \quad F_n(t) = -\frac{1}{\mu_n^3}F_n^{(3)}(t), \tag{24}$$

where

$$\psi_n^{(5)} = \sqrt{\frac{2}{l}}\int\limits_0^l \psi^{(5)}(x)\cos\mu_n x dx, \qquad \varphi_n^{(3)} = \sqrt{\frac{2}{l}}\int\limits_0^l \varphi^{(3)}(x)\cos\mu_n x dx,$$

$$\varphi_n^{(7)} = \sqrt{\frac{2}{l}}\int\limits_0^l \varphi^{(7)}(x)\cos\mu_n x dx, \qquad F_n^{(3)}(t) = \sqrt{\frac{2}{l}}\int\limits_0^l F_{xxx}(x,t)\cos\mu_n x dx$$

with the estimates

$$\sum_{n=1}^{\infty} |\psi_n^{(5)}|^2 \leqslant \|\psi^{(5)}(x)\|_{L_2[0,l]}, \qquad \sum_{n=1}^{\infty} |\varphi_n^{(3)}|^2 \leqslant \|\varphi^{(3)}(x)\|_{L_2[0,l]}$$

$$\sum_{n=1}^{\infty} \left|\varphi_n^{(7)}\right|^2 \leqslant \left\|\varphi^{(7)}(x)\right\|_{L_2[0,l]}, \qquad \sum_{n=1}^{\infty} |F_n^{(3)}(t)|^2 \leqslant \|F_{xxx}(x,t)\|_{C([0,T],L_2[0,l])}. \tag{25}$$

We integrate Equation (16) by parts several times: the integral having integrate functions $\varphi(x)$, $\varphi^{(4)}(x)$ and $f(x,t)$ - three times, the integral having integrate functions $\psi(x)$ - five times. Given the conditions of Lemma 2.2, we obtain the equalities (24). Inequality (25) is the Bessel inequality for the coefficients of the Fourier expansions of the functions $\psi_n^{(5)}$, $\varphi_n^{(3)}$, $\varphi_n^{(7)}$, and $F_n^{(3)}(t)$ in the cosine system $\{\sqrt{2/l}\cos\mu_n x\}$ on the interval $[0,l]$.

Note that if the conditions of Lemma 2.2 are fulfilled, the series in (23) converge. Let us show this for the first series, for the other series it is quite similar. Indeed, using (24) and the Cauchy-Bunyakovsky inequality, we have

$$\sum_{n=1}^{\infty} n^4|\psi_n| = \left(\frac{l}{\pi}\right)^5 \sum_{n=1}^{\infty} \frac{1}{n}|\psi_n^{(5)}| \leq \left(\frac{l}{\pi}\right)^5 \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}|\psi_n^{(5)}|^2},$$

Hence, according to the second inequality in (25), the convergence of the numerical series follows $\sum\limits_{n=1}^{\infty} n^4|\psi_n|$.

If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 2.2, then, due to of (24) and (25), the series in (23) converge, and consequently the series (10), (11) and (12) converge absolutely and uniformly in $\overline{D}$. Thus, the sum of the series (10) satisfies the relations (6)-(8).

Thus, we have proven the following statement.

Theorem 2.2. *Let $k(t) \in C[0,T]$. If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ satisfy the conditions (A1)-(A3), then there exists a unique classical solution to the problem* (6)-(8).

It is clear that, based on the found function $v(x,t)$, the solution to the problem (1)–(3) is determined by the formula

$$u(x,t) = \varphi(x) + \int_0^t v(x,\tau)d\tau.$$

Now we will prove the following statement, which will be used in the study of the inverse problem.

Lemma 2.3. *Let v_n and $\tilde{v}_n$ be two solutions of equation* (16) *corresponding to different k, $\tilde{k}$, with the same data φ, ϕ, ψ and f. Then the following estimate holds:*

$$|v_n(t) - \tilde{v}_n(t)| \leqslant d_n\|k - \tilde{k}\|, \quad t \in [0,T], \tag{26}$$

where

$$d_n := \frac{1}{a\mu_n^2}\left[(1+T^2C_1)|\psi_n| + \frac{1}{n^2}|\phi_n| + \frac{1}{n^2}|\varphi_n| + \frac{1}{n^2}\|F_n\|\right]\exp\left\{\frac{\|k\|T^2}{2a\mu_1^2}\right\}.$$

Proof. Since v_n, $\tilde{v}_n$ are two solutions of equation (16), corresponding to the functions k, $\tilde{k}$, then for the modulus of the difference of these functions the following estimate holds:

$$|v_n(t)-\tilde{v}_n(t)| \leqslant \frac{1}{a\mu_n^2}|\varphi_n| \int_0^t \left|k(\tau)-\tilde{k}(\tau)\right| d\tau$$
$$+\frac{1}{a\mu_n^2}\int\limits_0^t \int\limits_\tau^t \left[|k(s-\tau)(v_n(\tau)-\tilde{v}_n(\tau))| + \left|\tilde{v}_n(\tau)(k(s-\tau)-\tilde{k}(s-\tau)\right|\right] ds d\tau.$$

Estimating each term on the right-hand side of this inequality and using formula (20), we obtain

$$\begin{aligned} |v_n(t)-\tilde{v}_n(t)| &\leqslant \frac{1}{a\mu_n^2}\left[(1+T^2C_1)|\psi_n| + \frac{1}{n^2}|\phi_n| + \frac{1}{n^2}|\varphi_n| + \frac{1}{n^2}\|F_n\|\right]\|k-\tilde{k}\| \\ &+ \frac{\|k\|T}{a\mu_n^2}\int\limits_0^t |v_n(\tau)-\tilde{v}_n(\tau)|\, d\tau. \end{aligned}$$

From the last inequality, according to Gronoull's lemma, the estimate (26) follows. □

2.2 Study of the inverse problem

The main finding of this paper is the following:

Theorem 2.3. *If the conditions(A1)-(A4) are satisfied, then for any $T>0$ on the interval $[0,T]$ there exists a unique solution to the inverse problem (6)–(9) of the class $k(t)\in C[0,T]$.*

Proof. Putting in (10) $x = x_0$, using (16) and using the additional condition (9), obtained by differentiating twice by t and after some simple transformations, we obtain the following integral equation with respect to $k(t)$:

$$k(t) = k_0(t) + \int\limits_0^t R(t-\tau)k(\tau)d\tau, \tag{27}$$

where

$$k_0(t) = \varphi^{-1}(x_0)\left[g^{(3)}(t) + \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left[\psi_n a^2\mu_n^4 \cos a\mu_n^2 t - a^3\mu_n^4\varphi_n^{(4)}\sin a\mu_n^2 t \right.\right.$$
$$\left.\left. - F_n(t) + a\mu_n^2 \int\limits_0^t F_n(s)\sin a\mu_n^2(t-s)ds + \mu_n^2\varphi_n\left(1-\cos a\mu_n^2 t\right)\right]\sin\mu_n x_0\right],$$

$$R(t-\tau) = -\varphi^{-1}(x_0)\left[g'(t-\tau) - a\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^2\int\limits_\tau^t v_n(s-\tau)\sin a\mu_n^2(t-s)ds\sin\mu_n x_0\right].$$

The solution of integral equation (16) depends on $k(t)$. Therefore, in the future, we will put $v_n = v_n(t;k)$ in $R(t-\tau)$.

We introduce the operator A, defining it on the right-hand side (27). Then, the equation (27) is rewritten in a more compact form:

$$k(t) = A[k](t). \tag{28}$$

Let us denote by $C_\rho[0,T]$ the space of continuous functions with finite norm

$$\|k\|_\rho = \max_{t\in[0,T]} \left|k(t)e^{-\rho t}\right|, \tag{29}$$

$\rho > 0$ is some fixed number. Due to the fact that $e^{-\rho T}\|k\| \leqslant \|k\|_\rho \leqslant \|k\|$ the norms $\|k\|_\rho$ and $\|k\|$ are equivalent for any fixed $T \in (0,\infty)$. We will choose the number ρ later. Let

$$S_\rho(k_0,\alpha) := \left\{k(t):\ k(t) \in C[0,T],\ \|k-k_0\|_\rho \leqslant \alpha\right\}.$$

It is not difficult to see that for $k \in S_\rho(k_0,\alpha)$ there is an estimate of

$$\|k\|_\rho \leqslant \|k_0\|_\rho + \alpha \leqslant \|k_0\| + \alpha := \alpha_0, \tag{30}$$

so α_0 is a known number.

Further, it can be demonstrated that for a suitable choice of $\rho > 0$ the operator A realizes contracted mapping of the ball $S_\rho(k_0,\alpha)$ onto itself; i.e., the condition $k(t) \in S_\rho(k_0,\alpha)$ implies that $A[k](t) \in S_\rho(k_0,\alpha)$ and A compresses the distance between any elements $\left\{k(t),\tilde{k}(t)\right\} \in S_\rho(k_0,\alpha)$.

Indeed, based on (28)-(30) and (26), after some calculations we have estimates

$$\|A[k](t) - k_0(t)\|_\rho = \max_{t\in[0,T]}\left|\int_0^t R(t-\tau)k(\tau)e^{-\rho\tau}e^{-\rho(t-\tau)}d\tau\right|$$

$$\leqslant \frac{1}{\varphi(x_0)}\left[\|g'\|T + \frac{aT^2}{2}\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^2 C_n \exp\left\{\frac{\alpha_0 T^2}{2a\mu_1^2}\right\}\right]\frac{\alpha_0}{\rho} =: \frac{1}{\rho}\beta_1;$$

$$\|A[k](t) - A[\tilde{k}](t)\|_\rho = \max_{t\in[0,T]}\left|\left(A[k](t) - A[\tilde{k}](t)\right)e^{-\rho t}\right|$$

$$\leqslant \frac{1}{|\varphi(x_0)|}\left[\|g'\|T + \frac{aT^2}{\sqrt{2l}}\sum_{n=1}^{\infty}\mu_n^2\left[d_n\alpha_0 + C_n\exp\left\{\frac{\alpha_0 T^2}{2a\mu_1^2}\right\}\right]\right]\frac{\|k-\tilde{k}\|_\rho}{\rho} =: \frac{\beta_2}{\rho}\|k-\tilde{k}\|_\rho.$$

In these inequalities, we have also estimated in C_n and d_n the norm $\|k\|$ by α_0 according to (30). Note that all numerical series in them, taking into account the conditions of Theorem 2.3, are convergent.

Thus, as follows from the above estimates, if the number ρ is chosen from the condition $\rho > \max\{\alpha^{-1}\beta_1, \beta_2\}$, then the operator A is contractive on $S_\rho(k_0,\alpha)$. In this case, according to Banach's principle, the equation (28) has a unique solution in $S_\rho(k_0,\alpha)$ for any fixed $T > 0$. □

3 Numerical procedure

We divide the interval $(0,l)$ into discrete points using a total of N_x discretization points. The spatial step size is defined as $\Delta x = \frac{l}{N_x}$. The discretized points are represented as $x_i = i\Delta x$, where $0 \leqslant i \leqslant N_x$ and i is a positive integer. To approximate the solution

$u(x_i, t_n)$, we introduce the notation u_i^n, where $t_n = n\Delta t$ and $0 \leqslant n \leqslant N_t$. Here, $\Delta t = \frac{T}{N_t}$ represents the temporal step size, and N_t is the total number of time steps.

In this paper, we use forward difference method for the inverse problem. This method is conditionally stable and a sufficient condition for stability of the method is

$$\left(\frac{a\Delta t}{\Delta x^2}\right)^2 \leqslant 2.$$

Although we cannot claim what will happen if this condition is violated, it provides an upper bound for stability.

3.1 Fully explicit finite difference scheme

To determine the unknown coefficient $k(t)$, we employ the finite difference method on the equivalent form presented in equations (6) – (9), which represent the inverse problem defined in equations (1) – (3). We adopt the same procedure as in [16], however, to simplify the scheme, we utilize a three-point stencil rather than the five-point stencil employed in [16]. This simplification involves denoting the second derivative as $w = v_{xx}$ with its approximation given by:

$$w_i \approx \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2}.$$

Employing central second-order approximations for both temporal and spatial derivatives as

$$v_{tt} \approx \frac{v_i^{n+1} - 2v_i^n + v_i^{n-1}}{\Delta t^2}, \quad w_{xx} \approx \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\Delta x^2}.$$

Right-hand side of Eq. (6) can be rewritten as:

$$k(t)\phi(x) + \int_0^t k(\tau)v(x, t-\tau)d\tau + F(x,t) \approx k^n\phi_i + \Delta t \sum_{j=1}^{n} k^j v_i^{n-j} + F_i^n.$$

The integral itself is approximated using a rectangular integration technique. Then, fully explicit finite difference is applied,

$$v_i^{n+1} = 2v_i^n - v_i^{n-1} + \Delta t^2 \left(-a^2 w_i^{(2)n} + k^n\phi_i + \Delta t \sum_{j=1}^{n} k^j v_i^{n-j} + F_i^n\right), \tag{31}$$

where $w_i^{(2)n}$ is the numerical approximation of w_{xx}. The additional condition, $v(x_0, t) = G(t)$, is used to determine the unknown coefficient. Here, $G(t) = g'(t)$ and x_0 is within the range 0 to l. Let's denote its subscript as p. Now, we rewrite the above scheme for $i = p$, leading to

$$k^n\phi_p + \Delta t \sum_{j=1}^{n} k^j v_p^{n-j} = S^n, \quad \text{where} \quad S^n = \frac{G^{n+1} + 2G^n - G^{n-1}}{\Delta t^2} + a^2 w_p^{(2)n} - F_p^n.$$

The numerical scheme for finding k^n can be easily devised as

$$\begin{aligned}
&n = 0 \quad &&\rightarrow \quad k^0 = \frac{S^n}{\phi_p}, \\
&n = 1 \quad &&\rightarrow \quad k^1\phi_p + \Delta t k^1 v_p^0 = S^1 \quad \rightarrow \quad k^1 = \frac{S^1}{\phi_p + \Delta t v_p^0}, \\
&n = 2 \quad &&\rightarrow \quad k^2\phi_p + \Delta t(k^1 v_p^1 + k^2 v_p^0) = S^2 \quad \rightarrow \quad k^2 = \frac{S^2 - \Delta t k^1 v_p^1}{\phi_p + \Delta t v_p^0}.
\end{aligned}$$

Hence,

$$\begin{aligned} k^n &= \frac{S^n}{\phi_p} \quad \text{for} \quad n = 0 \\ k^n &= \frac{1}{\phi_p + \Delta t v_p^0} \left(S^n - \Delta t \sum_{j=1}^{n-1} k^j v_p^{n-j} \right) \quad \text{for} \quad n > 0. \end{aligned} \tag{32}$$

4 Numerical results

The inverse problem (6)–(9) can be solved numerically using the finite difference schemes (31) for $v(x,t)$ and (32) for $k(t)$. The results have been analyzed by calculating the absolute error between the exact and numerical solutions, defined as

$$\eta(v) = \max(v_i^{\text{numerical}} - v_i^{\text{exact}}), \quad i \in (0, N_x)$$

and

$$\eta(k) = \max(k_n^{\text{numerical}} - k_n^{\text{exact}}), \quad n \in (0, N_t).$$

The following test example is considered for the numerical accuracy check ($a = 2$)

$$F(x,t) = -18e^{-2t}\sin x + e^{-t}\sin x,$$

$$\varphi(x) = \sin x, \quad \psi(x) = -2\sin x, \quad \phi(x) = 4\sin x, \quad g(t) = \frac{\sqrt{2}}{2}e^{-2t},$$

for $x \in (0, l = 2\pi)$ and $t \in (0, T = 1)$, $G = g'(t)$ is determined analytically, i.e. $G = \sqrt{2}e^{-2t}$. The exact solution is given by

$$v(x,t) = -2e^{-2t}\sin x \quad \text{and} \quad k(t) = e^{-t}.$$

For the additional condition, we take $x_0 = \pi/4$.

Figs. 1 and 2 illustrate the numerical solution compared to the analytical solution for both $u(x,t)$ and $k(t)$ over the time interval 0 to 1, showing good agreement. We used $N_x = 128$ and $N_t = 5000$, resulting in absolute errors as shown in Fig. 3. While opting for a higher discretization points N_x could enhance accuracy, it would inevitably escalate computation time.

Introducing noise to an inverse problem can be advantageous as it serves to regularize the problem and mitigate overfitting. In the absence of noise, the solution to an inverse problem tends to be highly sensitive to minor variations in the input data, which may hinder its ability to generalize effectively to new or noisy data. By incorporating noise into the data, the inverse problem becomes more well-posed, implying that it possesses a unique and stable solution that is less susceptible to small perturbations in the input data. Therefore, the proposed algorithm is tested by adding noise to the additional condition, $v(x_0,t) = G(t)$, hence

$$v(x_0,t) = G(t)\left(1 + \frac{P}{100}\xi\right),$$

where P – error in percentage, ξ – a random number from a uniform distribution over $(-1,1)$. Figs. 4-6 presents the obtained numerical solutions of $k(t)$ with noise P set to 3%, 5%, and 7%. It is evident that the finite difference scheme performs satisfactorily as time progresses toward 1. Nevertheless, for higher noise levels, particularly when P equals 7, the fluctuation range is initially high but stabilizes as time increases.

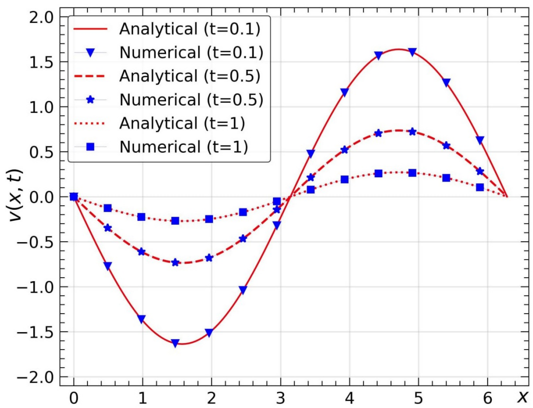

Figure 1: Comparison of the numerical and analytical solutions for $v(x,t)$.

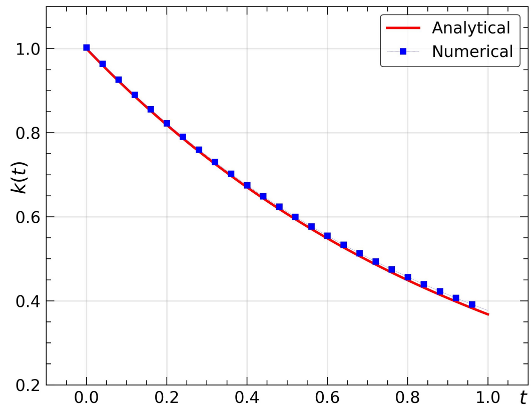

Figure 2: Comparison of the numerical and analytical solutions for $k(t)$.

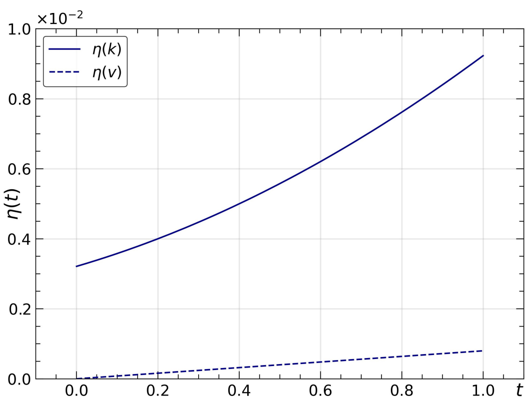

Figure 3: Estimated absolute errors $\eta(u)$ and $\eta(k)$.

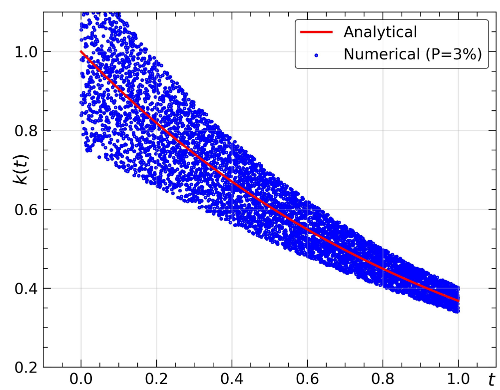

Figure 4: Numerical solution of $k(t)$ obtained with noise P=3%.

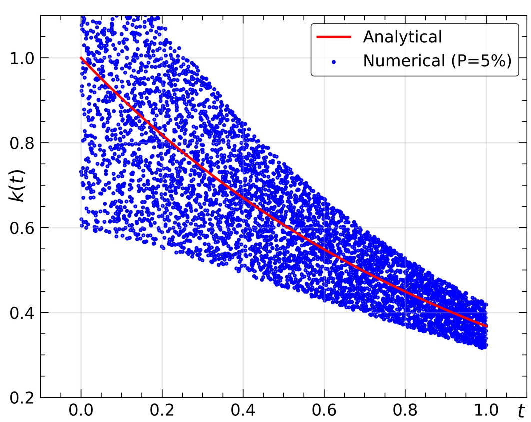

Figure 5: Numerical solution of $k(t)$ obtained with noise P=5%.

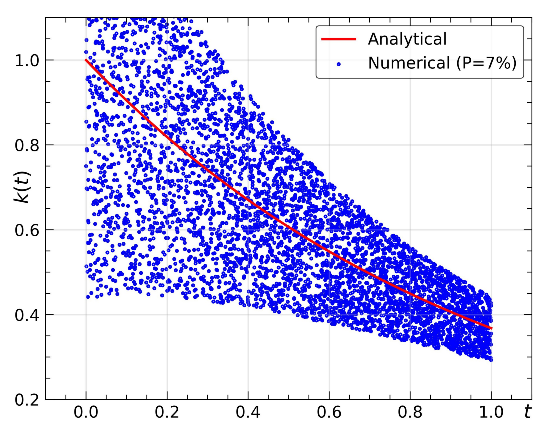

Figure 6: Numerical solution of $k(t)$ obtained with noise P=7%.

5 Conclusion

In conclusion, this work addresses the problem of identifying the kernel that represents the memory of the medium in the integro-differential equation of forced vibrations of a beam. The problem is reduced to integral equations of the second kind of Volterra type with respect to the solution of the direct problem and the unknown kernel of the inverse problem. To solve these equations, we apply the method of compressive mappings in the space of continuous functions with exponential weight norm. The inverse problem's global solvability and the solution's conditional stability are established.

In addtion, this paper presents an efficient numerical solution for the inverse problem of the beam equation using the finite difference method. The unknown coefficient in the equation is expressed in terms of an integral, allowing for its efficient determination alongside the solution for $u(x,t)$. Through rigorous algorithmic development, the numerical scheme offers accurate solutions for both $u(x,t)$ and the unknown coefficient $k(t)$. To validate the effectiveness of the proposed method, a comprehensive test example was carefully selected. The numerical results obtained were subsequently compared to analytical solutions, revealing a noteworthy level of agreement.

Furthermore, the performance of the proposed scheme was evaluated under noisy conditions. It was observed that while numerical fluctuations initially exhibit significant variance, they steadily diminish over time, demonstrating the scheme's robustness in handling noise and its ability to converge towards stable solutions as time evolves.

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